MMAT5520 Differential Equations & Linear Algebra
Final Exam (12 Dec 2012)
Time allowed: 120 mins

Each question carries 10 marks marks

1. Given that $y_1(t) = t$ is a solution to the equation $t^2y'' - ty' + y = 0$, $t > 0$.
   (a) Use the method of reduction of order to find a solution $y_2(t)$ such that $y_2(t)$ is not a linear function.
   (b) Explain why $y_1(t), y_2(t)$ are linearly independent.

2. Write down an appropriate form of a particular solution (do not solve the equation) to each of the following equations.
   (a) $y^{(3)} + 4y'' + 4y' = 4te^{3t} - te^{-2t} + \sin 2t$
   (b) $y^{(4)} + 2y'' + y = te^t \cos t + 3t^2 - 2t \cos t$

3. Let
$$\mathbf{A} = \begin{pmatrix} 0 & 2 & -3 \\ 2 & -3 & 6 \\ 2 & -4 & 7 \end{pmatrix}.$$
   (a) Find a non-singular matrix $\mathbf{P}$ which diagonalizes $\mathbf{A}$.
   (b) Find $\mathbf{A}^7$ in terms of $\mathbf{P}$ and $\mathbf{P}^{-1}$. (You don't need to find $\mathbf{P}^{-1}$.)
   (c) Express $\mathbf{A}^{-1}$ in the form $a\mathbf{A}^2 + b\mathbf{A} + c\mathbf{I}$ for real numbers a, b, c.

4. Solve the homogeneous systems $\mathbf{x}' = \mathbf{A}\mathbf{x}$, where derivatives are taken with respect to t, for the following matrix $\mathbf{A}$.
   (a) $\mathbf{A} = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix}$
   (b) $\mathbf{A} = \begin{pmatrix} -1 & -2 & 4 \\ 1 & 2 & -2 \\ 0 & 0 & 1 \end{pmatrix}$

5. Let $\mathbf{A} = \begin{pmatrix} 5 & 1 & -3 \\ -1 & 3 & 2 \\ 1 & 0 & 1 \end{pmatrix}$
   (a) Find a generalized eigenvector of rank 3 of $\mathbf{A}$.
   (b) Solve
$$\mathbf{x}' = \mathbf{A}\mathbf{x},$$
   where the derivative is taken with respect to t.

6. Let
$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}.$$

   (a) Find $\exp(\mathbf{A}t)$.

   (b) Find a fundamental matrix $\Psi(t)$ for the homogeneous system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ with $\Psi(0) = \begin{pmatrix} 1 & 1 \\ -2 & 0 \end{pmatrix}$.

   (c) Solve the non-homogeneous system
$$\mathbf{x}' = \mathbf{A}\mathbf{x} + 90 \begin{pmatrix} 2t \\ 1 \end{pmatrix}$$

7. Let $\mathbf{A}$ be a 3×3 matrix. Suppose $\mathbf{A}$ has an eigenvalue λ and a generalized eigenvector η of rank 3 associated with λ.

   (a) Explain whether $\mathbf{A}$ is diagonalizable.

   (b) Find the minimal polynomial of $\mathbf{A}$. Explain your answer.

   (c) Let $\eta_1 = (\mathbf{A} - \lambda\mathbf{I})\eta$ and $\eta_2 = (\mathbf{A} - \lambda\mathbf{I})^2\eta$. Prove that η, η_1, η_2 are linearly independent.

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